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THE DUALITY BETWEEN ASPLUND SPACES AND SPACES WITH THE RADON–NIKODYM PROPERTY

BY
CHARLES STEGALL

ABSTRACT
A Banach space X is an Asplund space (a strong differentiability space) if and only if X^* has the Radon–Nikodym property.

We give here a construction that, combined with results of [1, 4, 9, 10, 11], proves the following:

THEOREM 1. *Let X be a Banach space. Then X^* has the Radon–Nikodym Property (RNP) if and only if X is an Asplund space.*

A Banach space Y has RNP if for every probability space (S, Σ, μ), every $m: \Sigma \to Y$ that is countably additive, μ-continuous, of bounded variation, is representable by a Bochner integrable function. A function $f: S \to Y$ is Bochner integrable if it is Borel measurable, essentially separably valued, and $\int \| f(s) \| d\mu < +\infty$.

A Banach space Y is an Asplund space (what Asplund called a Strong Differentiability Space) if every continuous, real valued, convex function defined on an open subset of Y is Fréchet differentiable on a dense G_δ subset of its domain.

The main result we need is the following:

THEOREM 2. *Let X be a Banach space. Then X^* has RNP if and only if for every separable, linear subspace Y of X, Y^* is separable (in the norm topology).*

The sufficiency part of this theorem is basically due to R. S. Phillips (see [7]) and in the form stated here to Uhl [14]. The necessity part is due to the author [13].

Following [9] we shall say that a conjugate Banach space X^* is a DA space if

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every weak* compact, convex subset C of X^* is the weak* closed, convex hull of those points in C that are strongly exposed by some element of X. A point x in X strongly exposes C if there exists an x^* in C such that $x^*(x)$ is the supremum of x on C and if x_n^* in C and $x_n^*(x) \to x^*(x)$ then x_n^* converges in norm to x^*.

In [1] Asplund proved that if X is Asplund then X^* is a DA space. (This also follows from the main result of [13].) Recently, Namioka and Phelps [9] (also Collier [4]) have shown that if X^* is a DA space then X is an Asplund space.

In [11] (see also [2] for a stronger result) the following is proved: if, given any weak* compact, convex subset C of X^* with the property that for every $\varepsilon > 0$ there exists an x in X and a $t > 0$ such that

$$\{x^* \in C : x^*(x) > \sup_{x^* \in C} x^*(x) - t\}$$

has diameter less than ε, then X^* is a DA space. It follows easily from a result of Rieffel [12] that if X^* is a DA space then it has RNP. Thus we only need to prove the following:

PROPOSITION. *Suppose there exists a weak* compact, convex subset C of X^*, a $c > 0$, such that for every $x \in X$, every $t > 0$,*

$$\operatorname{diam}\{x^* \in C : x^*(x) > \sup_{x^* \in C} x^*(x) - t\} > c.$$

Then there exists a separable linear subspace Y of X such that Y^* is not norm separable.

PROOF. We shall assume that C is a subset of the unit ball of X^* and $0 < c < 1$. For $x \in X$, let

$$p(x) = \sup\{x^*(x) : x^* \in C\}$$

and for $t > 0$ let

$$S(x, t) = \{x^* \in C : x^*(x) > p(x) - t\}$$

be an open slice of C. Choose any x_0 in X and any t_0 such that $0 < t_0$. We shall show that there exist e_1^* and e_2^* in $S(x_0, t_0)$ that are extreme points of C and $\|e_1^* - e_2^*\| > c/4$. Choose e^* any extreme point of C in $S(x_0, t_0)$ (by the Krein–Milman theorem). Suppose all extreme points of C are contained in the union of $C \backslash S(x_0, t_0)$ and $B(x^*, c/4)$, the closed ball of radius $c/4$ and center x^*. Again, by the Krein–Milman theorem, since both sets are weak* compact and convex, each point in C is on a line segment running between them. Let $s = t_0 c/9$ and suppose $x^*(x_0) > p(x_0) - s$. If $x^* = u x_1^* + (1-u) x_2^*$ where $x_1^*(x_0) \leqq p(x_0) - t_0$ and $\|x_2^* - e^*\| \leqq c/4$. Then

$$\begin{aligned} p(x_0) - s < x^*(x_0) &= ux_1^*(x_0) + (1-u)x_2^*(x_0) \\ &\leqq u(p(x_0) - t_0) + (1-u)p(x_0) \\ &= p(x_0) - ut_0. \end{aligned}$$

Therefore, $u < s/t_0 = c/9$. So

$$\begin{aligned} \| x^* - e^* \| &\leqq u \| x_1^* - e^* \| + (1-u) \| x_2^* - e^* \| \\ &\leqq 2c/9 + c/4. \end{aligned}$$

This proves that the diameter of $S(x_0, s)$ is less than c, which is a contradiction. Suppose then that e_1^* and e_2^* are extreme points of C in $S(x_0, t_0)$. By the Hahn–Banach theorem, choose y_0 in X, $\| y_0 \| = 1$, such that $(e_1^* - e_2^*)(y_0) > c/4$. Let

$$U_1 = S(x_0, t_0) \cap S(y_0, p(y_0) - e_1^*(y_0) + c/12),$$

$$U_2 = S(x_0, t_0) \cap S(-y_0, p(-y_0) + e_2^*(y_0) + c/12).$$

Then $e_i^* \in U_i$ for $i = 1, 2$ and they are strongly extreme points of C (see [3]). There exists $y_{1,i} \in X$, $t_{1,i} > 0$ such that

$$e_i^* \in S(y_{1,i}, t_{1,i}) \subseteq U_i \quad \text{for } i = 1, 2.$$

Also, if $x_i^* \in U_i$ then

$$(x_1^* - x_2^*)(y_0) > c/12.$$

Repeat this construction inside both $S(y_{1,i}, t_{1,i})$ etc. obtaining

$$\{y_{n,i}\} \quad n = 1, 2, \cdots, \quad i = 1, 2, \cdots, 2^n;$$

$$t_{n,i} > 0;$$

$$\{x_{n,i}\} \quad n = 1, 2, \cdots, \quad i = 1, 2, \cdots, 2^n;$$

such that

(i) $S(y_{n+1,2i-j}, t_{n+1,2i-j}) \subseteq S(y_{n,i}, t_{n,i})$ for $j = 0, 1$;

(ii) $\sup\{x^*(x_{n,i}) \colon x^* \in S(y_{n+1,2i-1}, t_{n+1,2i-1})\} + c/12$

$$\leqq \inf\{x^*(x_{n,i}) \colon x^* \in S(y_{n+1,2i}, t_{n+1,2i})\}.$$

If we let Y be the smallest, closed, linear subspace of X containing $x_{n,i}$ then it is easy to see that

$$\left\{x^*\Big|_Y : x^* \in \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{2^n} \overline{S(y_{n,i}, t_{n,i})}\right\}$$

is a non norm separable subset of Y^*.

There are numerous corollaries to this result:

COROLLARY 1. *If X has an equivalent Fréchet differentiable norm or a C_1 function with bounded support then X is an Asplund space.*

It is easy to see using the Bishop–Phelps theorem and a result of Leduc (see [10]) that in either case X^* has RNP. Corollary 1 was also obtained in a different way by Ekeland and Lebourg [6].

If we replace Fréchet differentiable by Gateaux differentiable in the definition of Asplund spaces we obtain the weak-Asplund spaces. The following corollary is almost obvious.

COROLLARY 2. *Let X be an Asplund space. Suppose $T: X \to Y$ is a continuous linear operator whose range is dense in Y. Then Y is a weak-Asplund space.*

This corollary contains a classical result due to Mazur [8] (for any separable Banach space Y) and a result of Asplund for weakly compactly generated spaces [1] (using the factorization theorem of [5]).

REMARKS. By a more careful use of the results of [13], classical results on Fréchet differentiability, and a more complicated version of our Proposition one can prove a much more general result than Theorem 1. This will appear elsewhere.

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MATHEMATISCHES INSTITUT
BISMARCKSTRASSE $1\frac{1}{2}$
UNIVERSITÄT ERLANGEN–NÜRNBERG
D-8520 ERLANGEN, W. GERMANY